Soft vs. Hard Antenna Selection Based on the Minimum Distance for MIMO Systems

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Abstract

Assuming that Channel State Information (CSI) can be available at the transmitter, we provide a simplified representation of Multi-Input Multi-Output (MIMO) systems and derive a new precoder which maximizes the smallest distance between the received symbols. When the number of transmit antennas n_T is greater than the number of transmit data streams b, Heath and Paulraj have proposed in [2] a "hard" antenna selection (or switch precoder) which consists in choosing the best b (among n_T) antennas of the transmitter according to the criterion based on the minimum distance. Using the same criterion, we propose in this paper a "soft" (or linear) precoder that performs power allocation among the transmit antennas. Comparisons in term of Bit Error Rate (BER) between switch and linear precoders are performed considering $b = 2$ independent data streams , a QPSK modulation and the Maximum Likelihood (ML) receiver.

1 Introduction

Since a few years, it is well known that one way to get high rates on a scattering-rich wireless channel is to use multiple transmit and receive antennas [1]. Such systems are known as Multiple-Input Multiple-Output (MIMO) wireless. In some wireless applications feedback does exist and can be used to provide Channel State Information (CSI) to the transmitter. Performance can then be improved by the power allocation among eigenmodes thanks to the design of precoders which are based on an optimized diagonal representation of MIMO systems such as, for example, water filling (WF) [5], minimum mean square error (MMSE) [4], and minimum bit error rate (MBER) [3] solutions.

On the other hand, Heath and Paulraj have proposed in [2] a switch antenna selection based on the maximization of the smallest Euclidian distance $\max(d_{\min})$ between the noise-free received symbols. We propose in this paper a linear precoder using this criterion, which is particularly well adapted for the optimum ML receiver. The $\max(d_{\min})$ solution is possible thanks to the proposed simplified representation of MIMO systems and the precoder is derived in the case of two independent data streams. The resulting precoder gives a non-diagonal optimized MIMO system by opposition to power allocation strategies among eigen-modes (WF...).

The rest of the paper is organized as follows. The simplified representation of the MIMO system is presented in section 2. In section 3, both $\max(d_{\min})$ linear precoder and antenna selection are derived in the case of two virtual subchannels. Simulation results in term of BER are presented in section 4 and section 5 contains concluding remarks.

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2 MIMO channel simplified representation

Consider a MIMO system with n_R receive and n_T transmit antennas over which we want to achieve b independent data streams. For a MIMO channel without delay spread and including a precoder matrix $\mathbf{F}$ and a decoder matrix $\mathbf{G}$, the basic system model is:

$$\mathbf{y} = \mathbf{GHFs} + \mathbf{Gn} \tag{1}$$

where $\mathbf{H}$ is the $n_R \times n_T$ channel matrix, $\mathbf{F}$ is the $n_T \times b$ precoder matrix, $\mathbf{G}$ is the $b \times n_R$ decoder matrix, $\mathbf{s}$ is the $b \times 1$ transmitted vector and $\mathbf{n}$ is the $n_R \times 1$ noise vector.

step	i	method	$\mathbf{F}_i$	$\mathbf{G}_i$	$\mathbf{H}_{v_i}$
noise whitening	1	EVD: $\mathbf{R} = \mathbf{Q}_1\mathbf{\Lambda}_1\mathbf{Q}_1^*$	$\mathbf{F}_1 = \mathbf{I}_{n_T}$	$\mathbf{G}_1 = \mathbf{\Lambda}_1^{\frac{1}{2}}\mathbf{Q}_1^*$	$\mathbf{H}_{v1} = \mathbf{G}_1\mathbf{H}\mathbf{F}_1$
channel diagonalization	2	SVD: $\mathbf{H}_{v1} = \mathbf{A}_2\mathbf{\Sigma}_2\mathbf{B}_2^*$	$\mathbf{F}_2 = \mathbf{B}_2$	$\mathbf{G}_2 = \mathbf{A}_2^*$	$\mathbf{H}_{v2} = \mathbf{\Sigma}_2$
dimensionality reduction	3	$\mathbf{\Sigma}_2 = \begin{pmatrix} \mathbf{\Sigma}_r & 0 \\ 0 & 0 \end{pmatrix}$	$\mathbf{F}_3 = \begin{pmatrix} \mathbf{I}_b \\ 0 \end{pmatrix}$	$\mathbf{G}_3 = \begin{pmatrix} \mathbf{I}_b & 0 \end{pmatrix}$	$\mathbf{H}_v = \mathbf{G}_3\mathbf{H}_{v2}\mathbf{F}_3 = \mathbf{\Sigma}_b = diag\{\sqrt{\rho_1}, \ldots, \sqrt{\rho_b}\}$

Table 1: Steps to obtain the diagonal MIMO system in case of CSI at the transmitter.

We assume that $b \leq r = rank(\mathbf{H}) \leq \min(n_T, n_R)$ and $E\{\mathbf{ss}^*\} = \mathbf{I}_b$, $E\{\mathbf{nn}^*\} = \mathbf{R}$ and[1] $E\{\mathbf{sn}^*\} = \mathbf{0}$. Furthermore, if the available transmission power is noted p_0, the constraint below must be fulfilled: $trace\{\mathbf{FF}^*\} = p_0$.

Before the optimization, the first objective is to obtain a diagonal channel and a whitened noise in order to facilitate both the system analysis and the determination of the optimal precoder. Our approach is based on the decomposition of the precoder and decoder matrices $\mathbf{F} = \mathbf{F}_v\mathbf{F}_L$ and $\mathbf{G} = \mathbf{G}_L\mathbf{G}_v$.

The model (1) then becomes:

$$\mathbf{y} = \mathbf{G}_L\mathbf{H}_v\mathbf{F}_L\mathbf{s} + \mathbf{G}_L\mathbf{n}_v \qquad (2)$$

where $\mathbf{H}_v = \mathbf{G}_v\mathbf{H}\mathbf{F}_v$ is the virtual channel and $\mathbf{n}_v = \mathbf{G}_v\mathbf{n}$ is the virtual noise with correlation matrix $\mathbf{R}_v = \mathbf{G}_v\mathbf{R}\mathbf{G}_v^*$. We will only use virtual precoder matrices $\mathbf{F}_v$ with orthonormal columns, and then $\mathbf{F}_v^*\mathbf{F}_v = \mathbf{I}$. As a consequence, the power constraint becomes $trace\{\mathbf{F}_L\mathbf{F}_L^*\} = p_0$. Furthermore, as an ML receiver is used we can consider $\mathbf{G}_L = \mathbf{I}_b$.

The simplified system will be obtained via successive transformations, so the virtual precoder and decoder matrices will be defined as $\mathbf{G}_v = \mathbf{G}_3\mathbf{G}_2\mathbf{G}_1$ and $\mathbf{F}_v = \mathbf{F}_1\mathbf{F}_2\mathbf{F}_3$. The different steps to obtain a diagonal MIMO channel are summarized in Tab. 1. This simplified model is true whatever the number of antennas and the constellation are. Note that the diagonal entries $\sqrt{\rho_i}$ for $i = 1, \ldots b$ of $\mathbf{H}_v$ are sorted in a decreasing order and ρ_i corresponds to the SNR of the i^{th} virtual subchannel. Otherwise, the noise statistic after each step is circular complex Gaussian with correlation matrices: $\mathbf{R}_{v1} = E[\mathbf{n}_{v1}\mathbf{n}_{v1}^*] = \mathbf{I}_{n_R}$, $\mathbf{R}_{v2} = E[\mathbf{n}_{v2}\mathbf{n}_{v2}^*] = \mathbf{I}_{n_R}$ and $\mathbf{R}_v = E[\mathbf{n}_v\mathbf{n}_v^*] = \mathbf{I}_b$.

[1] the superscript * denotes the conjugate transposition

3 Minimum Euclidian distance precoder

3.1 Switch precoder

The principle of MIMO transmission using a switch precoder [2] for antenna selection is illustrated in Fig. 1.

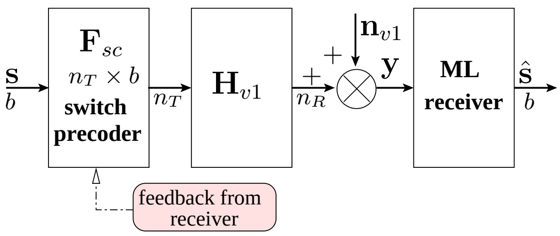

Figure 1: MIMO equivalent transmission system with a switch precoder

On the receiver side only a noise whitening operation is performed (see step 1 in Tab. 1), which leads to a whitened noise vector $\mathbf{n}_{v1}$ and an equivalent channel matrix $\mathbf{H}_{v1}$ (non-diagonal). On the transmitter side a switch matrix $\mathbf{F}_{sc}$ allows to select b antennas among n_T according to a criterion when feedback from the receiver exists. For example if we consider a system with $n_T = 4$ and $b = 2$, the following matrix $\mathbf{F}_{sc}$ selects the second and the third antenna:

$$\mathbf{F}_{sc} = \sqrt{p_0/2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The system equation is:

$$\mathbf{y} = \mathbf{H}_{v1}\mathbf{F}_{sc}\,\mathbf{s} + \mathbf{n}_{v1} \qquad (3)$$

The minimum Euclidian distance denoted d_{min} is given by:

$$d_{min}^2 = \min_{\mathbf{s}_i, \mathbf{s}_j \in \mathcal{S},\, \mathbf{s}_i \neq \mathbf{s}_j} \|\mathbf{H}_{v1}\mathbf{F}_{sc}(\mathbf{s}_i \quad \mathbf{s}_j)\|^2. \qquad (4)$$

where $\mathcal{S}$ is the set of all possible transmitted vector $\mathbf{s}$. The largest d_{min} is determined by choosing b antennas among n_T:

$$d_{sc} = \max_{F_{sc}}(d_{min}). \quad (5)$$

3.2 Linear precoder

The principle of MIMO transmission using a linear precoder is illustrated in Fig. 2.

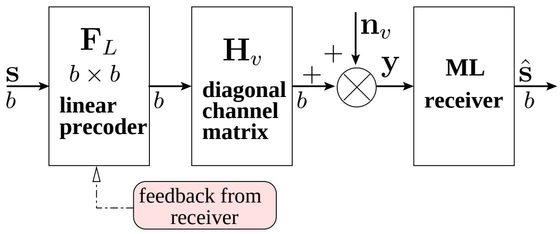

Figure 2: MIMO equivalent transmission system with a linear precoder.

From the results established in section 2 and Tab. 1 on the MIMO simplified representation, a linear precoder matrix $\mathbf{F}_L$ remains to be determined according to the $\max(d_{min})$ criterion. The system equation is:

$$\mathbf{y} = \mathbf{H}_v \mathbf{F}_L \, \mathbf{s} + \mathbf{n}_v \quad (6)$$

Note that the dimensional reduction (third step in Tab. 1) is already performed but physically the $n_T \geq b$ transmit antennas are used by opposition to the switch antenna selection presented in subsection 3.1. However these two systems (switch and linear precoders) have the same spectral efficiency and can be compared in order to determine the BER improvement.

In the following, the results to derive the linear $\max(d_{\min})$ precoder are obtained only for $b = 2$. The generalization for $b > 2$ is not straightforward. A QPSK modulation is considered in the paper. Other modulations can be considered by following the proposed approach but the minimum distances presented further must be re-evaluated in this case.

Let us denote $\mathbf{F}_L = \begin{pmatrix} x & y \\ w & z \end{pmatrix}$. In order to make the search easier, we consider the elements of $\mathbf{F}_L$ as real. Complex values imply rotations on the received constellation defined as $\{\mathbf{H}_v \mathbf{F}_L \, \mathbf{s} \mid \mathbf{s} = [s_1 \quad s_2]^T \in \mathcal{S}\}$. So the proposed solution is not strictly optimal in term of $d_{\min}$ due to this real-valued restriction.

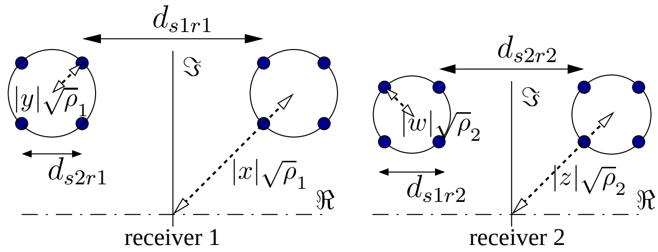

Figure 3: Upper part of the noise-free received constellation for QPSK (with $|x| > 2|y|$ and $|z| > 2|w|$)

From Fig. 3, the minimum distance d_{s_1} and d_{s_2} for the symbol s_1 and s_2 are respectively given by:

$$d_{s_1}^2 = d_{s1r1}^2 + d_{s1r2}^2 = 2\rho_1(|x| \quad |y|)^2 + 2\rho_2|w|^2 \quad (7)$$
$$d_{s_2}^2 = d_{s2r1}^2 + d_{s2r2}^2 = 2\rho_1|y|^2 + 2\rho_2(|z| \quad |w|)^2 \quad (8)$$

with $|x| > |y|$ and $|z| > |w|$. The optimized minimum distance is obtained for $d_{min} = d_{s_1} = d_{s_2}$. The search domain for $|x|/|y| \in]1/2, 2[$ is discarded due to a loss of power. In fact, as we can always find for $|x|/|y| \notin]1/2, 2[$ the same minimum distances on the first receiver but with less power, we can then restrict the search domain to $|x| > 2|y|$ which implies $|z| > 2|w|$ (thanks to the optimized condition $d_{s_1} = d_{s_2}$). This condition does not induce any loss of generality because the alternative condition $|y| > 2|x|$ and $|w| > 2|z|$ corresponds to a permutation of the columns of $\mathbf{F}_L$ and gives the same $d_{\min}$.

In order to obtain by a simple way the precoder $\mathbf{F}_L$, we perform an Singular Value Decomposition (SVD):

$$\mathbf{F}_L = \mathbf{A}\mathbf{\Sigma}\mathbf{B}^* \quad (9)$$

where $\mathbf{A}$ and $\mathbf{B}$ are unitary matrices and $\mathbf{\Sigma}$ is a diagonal matrix with positive and decreasing ordered elements.

The optimization of $\mathbf{F}_L$ can be performed in two steps:

- $\mathbf{A}\mathbf{\Sigma}$ allows to choose the Singular Values (SV) of $\mathbf{H}_v\mathbf{F}_L$ because $\mathbf{B}^*$ has no impact on the SV. So our interest is to choose the largest ones. We prove in appendix A that the best choice for $\mathbf{A}$ is the identity matrix because it gives the largest SV of $\mathbf{H}_v\mathbf{F}_L$ for a given matrix $\mathbf{\Sigma}$.

- for the largest SV (*i.e.* $\mathbf{A} = \mathbf{I}$), we look for the matrices $\mathbf{B}$ and $\mathbf{\Sigma}$ which optimize d_{min}. As $\mathbf{F}_L$ is a real-valued matrix, $\mathbf{B}^*$ is necessarily a real-valued unitary matrix (*i.e.*, matrix of rotation) and the new expression of the linear precoder can be then expressed as: $\mathbf{F}_L = \mathbf{\Sigma}\mathbf{B}^*$

with $\mathbf{\Sigma} = \sqrt{p_0}\begin{pmatrix} \cos\psi & 0 \\ 0 & \sin\psi \end{pmatrix}$ and $\mathbf{B}^* = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. The search of ψ and θ can be restricted to $\psi \in [0, \pi/4[$ (to satisfy the decreasing order of the SV) and $\theta \in [0, \theta_{\max}]$ with $\tan\theta_{\max} = 1/2$. This last condition corresponds to the search domain $|x| \geq 2|y|$ and $|z| \geq 2|w|$.

The optimization of d_{min} is obtained for $d_{s_1} = d_{s_2}$. By substituting trigonometric notations in (7) and (8) the equality of the distances gives two solutions *i)* $\cos\theta = 2\sin\theta$ and *ii)* $\tan^2\psi = k$ with $k = \rho_1/\rho_2 > 1$.

i) the minimum square distance $d_{s_1}^2$ is equal to $2p_0\rho_2((k\ \ 1)\cos^2\psi + 1)/5$. The value of ψ which maximizes $d_{s_1}^2$ is obviously $\psi = 0$. The optimum precoder $\mathbf{F}_Q$ and its minimum square distance are then:

$$\mathbf{F}_Q = \sqrt{\frac{p_0}{5}}\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad d_Q^2 = 2p_0\rho_1/5 \quad (10)$$

One should note that the second row of $\mathbf{F}_Q$ is null, which means that the worst virtual subchannel is dropped but physically both the transmitter and the receiver do use all antennas. In fact, the precoder transforms both subchannel QPSK constellations into a received 16-QAM constellation on the most favored subchannel (the first here because elements of $\mathbf{H}_v$ are in the decreasing order).

ii) the minimum square distance $d_{s_1}^2$ can be expressed as: $d_{s_1}^2 = 2p_0\rho_1\rho_2/(\rho_1+\rho_2)(1\ \ 2\cos\theta\sin\theta + \sin^2\theta)$. $d_{s_1}^2$ is a decreasing function for $\theta \in [0, \theta_{max}]$, so the largest value is obtained for $\theta = 0$. The optimum precoder $\mathbf{F}_D$ and the corresponding minimum square distance are then in this case:

$$\mathbf{F}_D = \sqrt{\frac{p_0}{1+k}}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{k} \end{pmatrix} \text{ and } d_D^2 = \frac{2p_0\rho_1\rho_2}{\rho_1+\rho_2} \quad (11)$$

This precoder is diagonal, so it keeps the diagonal structure of the MIMO representation (cf section 2). We obtain on each receiver a QPSK constellation. This solution decouples the MIMO channel into 2 parallel and independent subchannels, which allows to apply the ML receiver separately on each subchannel. This solution is equivalent to the eigen-power allocation of the *equal-error* solution in [4] because (11) can be expressed in an equivalent form as:

$$\mathbf{F}_D = \frac{d_D}{\sqrt{2}}\begin{pmatrix} 1/\sqrt{\rho_1} & 0 \\ 0 & 1/\sqrt{\rho_2} \end{pmatrix} \quad (12)$$

and the output subchannel SNRs are given by: $SNR = SNR_1 = SNR_2 = d_D^2/2 = p_0\rho_1\rho_2/(\rho_1+\rho_2)$.

Finally, the optimal $d_{\min}$ precoder $\mathbf{F}_L$ for a QPSK modulation has two different simple expressions which depend on the ratio of virtual subchannels SNRs (*i.e.*, $k = \rho_1/\rho_2$). Fig. 4 plots d_D and d_Q versus k in order to choose the best precoder. For $k \leq 4$, the precoder is the diagonal one (*i.e.* $\mathbf{F}_L = \mathbf{F}_D$) and for $k \geq 4$, the precoder is $\mathbf{F}_L = \mathbf{F}_Q$. The threshold value of $k = 4$ is obtained by evaluating $d_Q = d_D$.

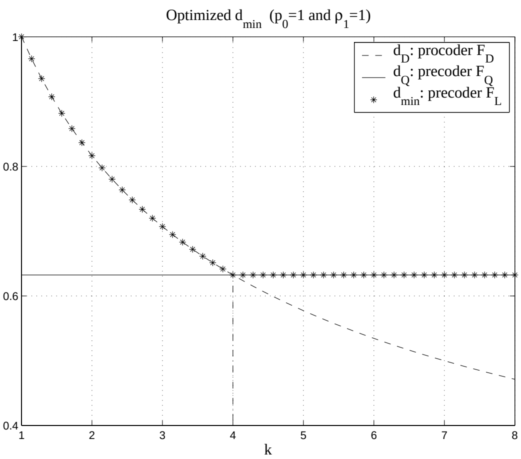

Figure 4: Optimized d_{min} for the precoders $\mathbf{F}_Q$ and $\mathbf{F}_D$.

4 Simulation results

The configuration is $n_T = 3$ transmitters, $n_R = 2$ receivers, and we transmit $b = 2$ independent data streams over the system. A QPSK constellation is used. For each SNR, 20 000 packets (a packet is 100 vectors of 2 symbols) are transmitted during the Monte-Carlo simulations. For each packet, a new $\mathbf{H}$ and a new $\mathbf{R}$ are randomly chosen, in order to obtain results that do not depend on a particular channel, nor on particular noise statistics. Entries of $\mathbf{H}$ are i.i.d. zero-mean unit-variance complex Gaussian random variables. Matrices $\mathbf{R}$ are obtained by $\mathbf{R} = \mathbf{T}\mathbf{T}^*$ (where entries of $\mathbf{T}$ are i.i.d. zero-mean unit-variance complex Gaussian random variables) and then scaled according to the desired SNR. The SNR is defined as the ratio of the total transmitted power *(i.e.*, p_0) to the total received noise power (*i.e.*, $trace(\mathbf{R})$). Performance improvement in term of BER of the proposed $\max(d_{\min})$ linear precoder $\mathbf{F}_L$ against the switch $\max(d_{\min})$ precoder $\mathbf{F}_{sc}$ is illustrated in Fig. 5. Note that performance of the $\max(d_{\min})$ linear precoder can be improved by substituting the MBER diagonal precoder[3] instead of the precoder F_D when the virtual subchannels SNR ratio is smaller than the threshold $k = 4$.

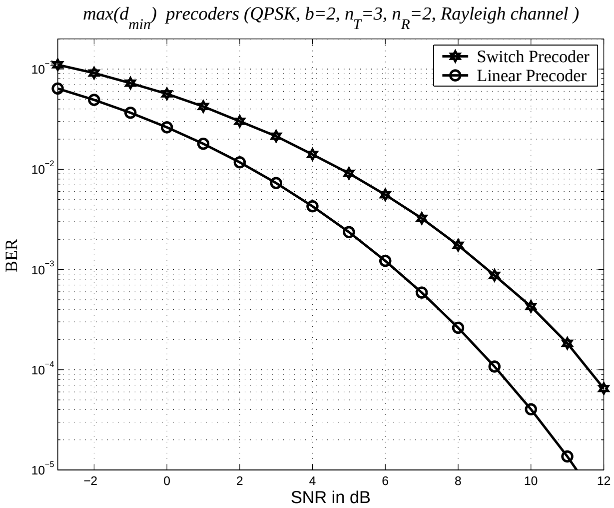

Figure 5: linear and switch $\max(d_{\min})$ precoders

5 Conclusion

We derived the linear precoder which maximizes the minimum Euclidian distance on the received constellations when $b = 2$ virtual independent subchannels are used. The $\max(d_{\min})$ solution for a QPSK modulation is particularly simple: the choice of the precoder $\mathbf{F}_D$ or $\mathbf{F}_Q$ depends on the ratio of the virtual subchannel SNRs. This kind of solution is new in comparison to the linear diagonal precoders that exist in the literature [4, 3], and its non-diagonality provides very good results in term of BER. Furthermore, the weighted antenna selection allows our $\max(d_{\min})$ soft precoder to outperform hard antenna selection proposed in [2].

A Proof matrix A is the identity

The general form of a unitary matrix $\mathbf{A}$ in dimension 2 is:

$$\mathbf{A} = \begin{pmatrix} \cos\phi\, e^{j\phi_1} & \sin\phi\, e^{j\phi_3} \\ \sin\phi\, e^{j\phi_2} & \cos\phi\, e^{j\phi_4} \end{pmatrix} \qquad (13)$$

under the constraint $(\phi_1 + \phi_3) \quad (\phi_2 + \phi_4) = k\pi$ with $\phi \in [0, \pi/2]$ and $\phi_i \in [0, 2\pi[$ for $i = 1, \ldots 4$.

The product $\lambda_1\lambda_2$ of the SV of $\mathbf{H}_v\mathbf{F}_L$ is equal to $det(\mathbf{H}_v\mathbf{A}\mathbf{\Sigma})$, then $\lambda_1\lambda_2 = det(\mathbf{H}_v\mathbf{\Sigma})det(\mathbf{A}) = det(\mathbf{H}_v\mathbf{\Sigma})$ which is independent of $\mathbf{A}$.

The sum $\lambda_1^2 + \lambda_2^2$ of the square SV is equal to $trace(\mathbf{H}_v\mathbf{A}\mathbf{\Sigma}^2\mathbf{A}^*\mathbf{H}_v) = \|\mathbf{H}_v\mathbf{A}\mathbf{\Sigma}\|^2$. Entries of $\mathbf{H}_v\mathbf{A}\mathbf{\Sigma} = [(\mathbf{H}_v\mathbf{A}\mathbf{\Sigma})_{ij}]$ are equal[2] to $\sqrt{\rho_i}\sigma_j a_{ij}$, then phases ϕ_i of $\mathbf{A}$ have no impact on $\lambda_1^2 + \lambda_2^2$.

Finally, from the two results above we can conclude that phases ϕ_i of $\mathbf{A}$ have no impact on λ_1 and λ_2. So, we can restrict to real-valued matrix:

$$\mathbf{A} = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \text{ with } \phi \in [0, \pi/2] \qquad (14)$$

[2] $\mathbf{A} = [a_{ij}]$ and $\mathbf{\Sigma} = diag\{\sigma_i\}$

In order to study the impact of ϕ on the SVD, Fig. 6 plots λ_1 vs. λ_2. The upper curve corresponds to $\phi = 0$ (i.e. $\mathbf{A} = \mathbf{I}$) for ψ between 0 and $\arctan(\sqrt{\rho_1/\rho_2}) < \pi/4$. The transversal curves are obtained for a fixed ψ and $\phi \in [0, \pi/2]$. Note that, as explained before, the product $\lambda_1\lambda_2$ is equal to $c = \sqrt{\rho_1\rho_2}\cos\psi\sin\psi$ and is independent of ϕ. So when ϕ varies from 0 to $\pi/2$, the SV start from the upper curve and follow a hyperbolic curve $\lambda_1 = c/\lambda_2$. In conclusion, our interest is to choose the largest SV for a fixed angle ψ by considering $\mathbf{A} = \mathbf{I}$.

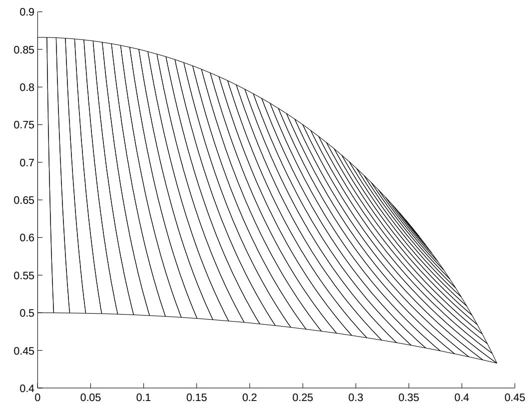

Figure 6: λ_1 versus λ_2 (with the virtual subchannel SNRs fixed to $\sqrt{\rho_1} = 0.866$ and $\sqrt{\rho_2} = 0.5$)

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