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# About the Reliability of Simulations of MIMO Transmission Systems

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*Abstract :* — The paper is concerned with the application of probability theory to Multi-Input Multi-Output (MIMO) transmission systems. We provide a theoretical formula to compute the error rate in such systems. Furthermore, analysis of this formula shows that simulation results are not reliable at high SNR. Finally, we provide an expression to compute the relative precision of the simulation results. This expression is helpful to know which simulation results are reliable and which are not.

*Key-Words :* — Wireless Digital Transmissions, MIMO, Error Rate, Random matrices, Simulation, Precision, Reliability.

## 1 Introduction

### 1.1 MIMO transmission systems

Recent research [2] has shown that very high spectral efficiency can be obtained over rich scattering wireless channels by using multielement antenna arrays at both transmitter and receiver (i.e. MIMO: Multi-Input, Multi-Output transmitters). The principle of MIMO transmission is as follows (Fig. 1): $n_T$ digital transmitters operate co-channel at a given symbol rate with synchronized symbol timing. $n_R$ digital receivers ($n_R \geq n_T$) also operate co-channel, with synchronized timing. On the receiver side, an optimal Maximum-Likelihood algorithm (or a suboptimal, but faster one, such as [3]) is used to estimate the transmitted symbols from the components of the received mixture.

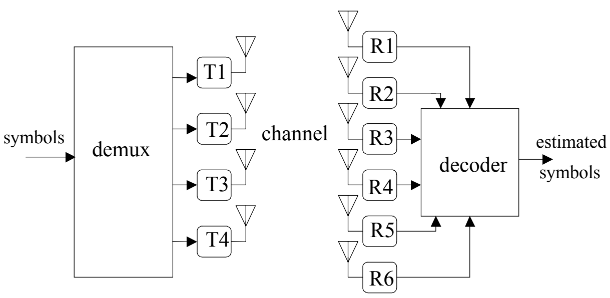


Figure 1: Principle of MIMO transmission

The MIMO channel is modelled by an $n_R \times n_T$ random matrix $H$, and the received vector $y$ (dimension $n_R$) is given by the equation below:

$$y = Hx + n \quad (1)$$

where $n$ is the noise vector (dimension $n_R$) and $x$ the transmitted vector (dimension $n_T$). The noise covariance matrix is $\sigma^2 I_{n_R}$.

Typical applications of MIMO systems are indoor (wireless local area networks) or urban mobile wireless communications. Using MIMO, spectral efficiencies far above the efficiency provided by single antenna transmission systems can be obtained (e.g. 20 bits/s/Hz [3]). The most widely used model for indoor or urban channels is the Rayleigh model [4]: the entries of $H$ are independent identically distributed circular complex gaussian random variables with zero mean and unit variance. Please note that considering unit variance does not imply any loss of generality because multiplying $y$ by any normalization constant does not modify the error rate. Similarly, the total transmit power is usually normalized to one (i.e. $E\left\{\|x\|^2\right\} = 1$). In this paper, we will also use these usual normalization hypotheses. Due to this normalization, the signal to noise ratio (SNR) in dB will be defined as $SNR = -20\log_{10}(\sigma)$.

In the sequel, **for illustration purpose**, the Figures provided correspond to results obtained for a MIMO system with $n_T = 2$ transmit antennas and $n_R = 3$ receive antennas, and BPSK (Binary Phase Shift Keying) signalling. However, the equations, the theory, and the conclusions **are not restricted to this particular case**.

## 1.2 Problems with simulation of MIMO transmission systems

A problem, that people in charge of designing MIMO systems are faced to, is the duration of simulations. In most work related in the literature, when a MIMO system is simulated in order to estimate its error characteristic (i.e. probability of error with respect to channel signal to noise ratio), the following Monte Carlo technique is used:

- $N_H$ channel matrices are randomly chosen
- For each channel matrix, $N_x$ vectors are successively transmitted (each vector is subject to random noise).
- The receiver estimates the transmitted vector and this is compared to the true transmitted vector.
- The number of false estimated vectors is used to estimate the probability of error.

The total number of transmitted vectors during the simulation is then $N = N_H N_x$. A reason for not choosing a new random channel matrix for each transmitted vector is that the receiver algorithms must be reoptimized for each new channel matrix (hence, adding more computational requirements). Another reason is that the channel is assumed almost stationary during the transmission of $N_x$ vectors.

Figure 2 (lower curve) shows a typical example of simulation results. The receiver is a maximum likelihood. The number of simulated channel matrices is $N_H = 100$ and for each channel matrix, $N_x = 2 \times 10^6$ vectors are transmitted. Hence, each point of the curve is obtained by estimating the probability of error from a total of $N = 2 \times 10^8$ transmitted vectors.

From probability theory, it is well known that, if the realizations are independent, the expected value of the number of errors is $P_e N$ and the standard deviation is $\sqrt{P_e(1-P_e)N}$. Hence, the relative precision, defined as the ratio between the standard deviation and the average value is:

$$\rho = \sqrt{\frac{(1-P_e)}{P_e N}} \qquad (2)$$

For $N = 2 \times 10^8$ and a probability of error equal to $10^{-6}$, we still have $\rho = 7\%$ . Hence, we could expect the lower curve on figure 2 to be a relatively good estimation of the MIMO system error characteristic. However, if we increase the number of simulated channel

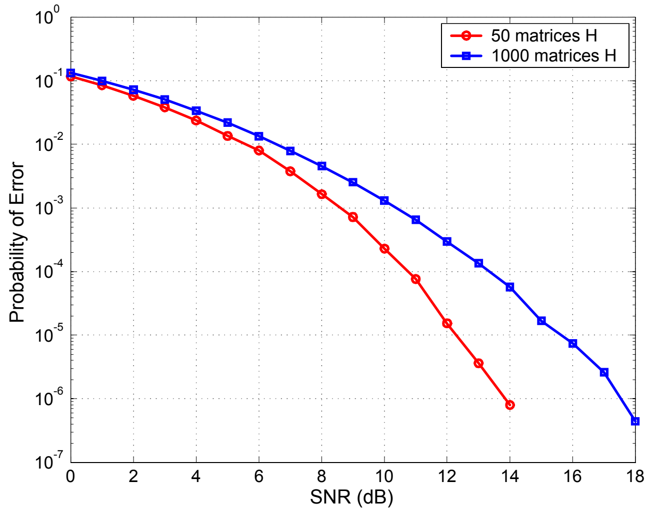


Figure 2: Impact of the number of channel matrices on simulation results

matrices by a factor of ten (i.e. $N_H = 1000$) and decrease $N_x$ by the same factor (i.e. $N_x = 2 \times 10^5$), in order to keep the same number of realizations $N = 2 \times 10^8$, we obtain the upper curve. The main observation is that, at high SNR, the estimated probability of error has considerably increased (by almost a factor 100, that is a 10000%, far above the 7% expected relative precision). Hence, the reliability of the obtained curves is highly questionable. Replaying the same experiment many times shows that the estimated error curve obtained with $N_H = 1000$ is "almost always" far above the estimated curve obtained with $N_H = 50$.

In fact, it is not difficult to guess that the reason for which the relative precision is not 7% as expected: as explained above, the same channel matrix is used for a set of $N_x$ successive transmit blocs, hence the $N$ realizations are not fully independent. But, far more difficult to explain is why increasing $N_H$ "almost always" increases the estimated probability of error.

## 1.3 Objective of the paper

In this paper, we use random matrices theory to derive a theoretical expression of the error rate. The probability of error we consider is the probability of vector error, that is the probability that the vector estimated by the Maximum Likelihood receiver is wrong. Thanks to this expression, some long and complex simulations could be avoided. Furthermore, the theoretical results derived in the sequel provide an explanation concerning the behavior of simulation outputs and allow to predict their true relative precision.

Application of random matrices theory to MIMO channels still represents a little part of scientific work concerning MIMO transmissions, and only a few recent works (such as [5] for theoretical prediction of capacity) have been published. The major part of work in the MIMO transmissions domain is dedicated to space-time coding, and receiver or precoder algorithms [6][7]. However, theoretical study of MIMO channels from the random matrices theory point of view is of crucial importance and can provide a lot of new results, because MIMO channels are basically multidimensional random channels.

The paper is organized as follows. In Section 2, we recall a few mathematical results about chi-square distributions and the Gamma function. In Section 3, we provide a theoretical formula of the error rate. Then, thanks to these results, we provide an expression concerning the reliability of simulations in Section 4. Finally, a conclusion is drawn in Section 5.

## 2 Mathematical recalls

The Gamma function $\Gamma(p)$ and the incomplete Gamma function $\Gamma_a(p)$ are defined below:

$$\Gamma(p) = \int_0^\infty t^{p-1}e^{-t}dt \quad (3)$$

$$= (p-1)! \qquad \text{(if } p \text{ is a positive integer)} \quad (4)$$

$$\Gamma_a(p) = \frac{1}{\Gamma(p)}\int_0^a t^{p-1}e^{-t}dt \quad (5)$$

$$= 1 - e^{-a}\sum_{k=0}^{p-1}\frac{a^k}{k!} \quad \text{(if } p \text{ is a positive integer)} \quad (6)$$

The probability density function (pdf) of the sum of the square moduli of $m$ i.i.d. circular complex gaussian random variables with zero mean and variance $\sigma_c^2$ is a chi-square distribution with $2m$ degrees of freedom:

$$p_c(t) = \frac{1}{\sigma_c^{2m}\Gamma(m)}t^{m-1}e^{-t/\sigma_c^2} \quad (7)$$

Its cumulative distribution function (cdf) is:

$$F_c(a) = \int_0^a p_c(t)dt \quad (8)$$

$$= \Gamma_{a/\sigma_c^2}(m) \quad (9)$$

## 3 Theoretical estimation of the probability of error

This Section is organized as follows. In Subsection 3.1, we explain why the minimum distance between noise-free received vectors is important to characterize the performances of a MIMO channel. Then, in Subsection 3.2, we derive the statistical distribution of the minimum distance, and in Subsection 3.3 we recall a theoretical result concerning the error rate for channels with a given minimum distance. Finally, in Subsection 3.4 we combine results from Subsections 3.2 and 3.3 to provide a theoretical expression of the error rate.

### 3.1 Definition of the minimum distance

Let us consider a MIMO transmission channel with $n_T$ transmit antennas and $n_R$ receive antennas ($n_R \geq n_T$), and let us note $M$ the number of symbols in the basic constellation (for instance, $M = 2$ for a BPSK modulation). The channel is modelled by an $n_R \times n_T$ random matrix $H$ mentioned in the introduction (see Eq. 1). We note $\mathcal{S} = \{s_p,\ p = 1, ..., M^{n_T}\}$ the set of all possible transmitted vectors (i.e. multidimensional constellation). As usual, the vectors $s_p$ are assumed to be normalized in order to have a total transmit power equal to 1. That is:

$$\frac{1}{M^{n_T}}\sum_{p=1}^{M^{n_T}}\left\|s_p\right\|^2 = 1 \quad (10)$$

For example, for a BPSK (Binary Phase Shift Keying) signalling and $n_T = 2$ transmitters, we have:

$$\mathcal{S} = \left\{\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\-1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\+1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}+1\\-1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}+1\\+1\end{pmatrix}\right\} \quad (11)$$

Let us note $d_0$ the minimum distance between the elements of $\mathcal{S}$. For instance, for the BPSK example mentioned above, we have $d_0 = \sqrt{2}$.

The noise-free received vectors belong to the set $\mathcal{R} = \{r_p,\ p = 1, ..., M^{n_T}\}$ where $r_p = Hs_p$. The probability of error is strongly linked to the minimum distance $d_{\min}$ between the elements of $\mathcal{R}$. Indeed, the maximum likelihood receiver searches for the element of $\mathcal{R}$ which is the closest to the actual received vector $y$. If $d_{\min}$ is small, some vectors of $\mathcal{R}$ are very close together and a small noise is sufficient to cause an error.

The minimum distance is:

$$d_{\min} = \min_{p\neq q}\left\|r_p - r_q\right\| \quad (12)$$

$$= \min_{p\neq q}\left\|H\left(s_p - s_q\right)\right\| \quad (13)$$

### 3.2 Cumulative distribution function of the minimum distance

A large value of $d_{\min}$ guarantees a low probability of error. Hence, knowing the statistical distribution of $d_{\min}$ is of great importance to characterize a MIMO transmission system and predict its error rate. In [1] we have shown that a good approximation is:

$$d_{\min} = d_0 \min_{m=1,\ldots,n_T} \|h_m\| \quad (14)$$

where $h_m$ is the $m^{th}$ column of $H$. This formula is important because it shows that the statistical distribution of $d_{\min}$ is determined by the statistical distribution of the norms of the columns of $H$. Since $\|h_m\|^2$ is the sum of $n_R$ square moduli of complex i.i.d. circular gaussian random variables with variances 1, it is chi-square distributed and, using equation 9, its cdf is:

$$F_h(u) = P\left(\|h_m\|^2 < u\right) \quad (15)$$
$$= \Gamma_u\left(n_R\right) \quad (16)$$

Hence, the cdf of $d_{\min}$ is:

$$F\left(a\right) = P\left(d_{\min} < a\right) \quad (17)$$
$$= 1 - \prod_{m=1}^{n_T} P\left(d_0^2 \|h_m\|^2 > a^2\right) \quad (18)$$
$$= 1 - \left(1 - F_h\left((a/d_0)^2\right)\right)^{n_T} \quad (19)$$

that is:

$$\boxed{F\left(a\right) = 1 - \left(1 - \Gamma_{(a/d_0)^2}\left(n_R\right)\right)^{n_T}} \quad (20)$$

where we recall that $d_0$ is the minimum distance between the possible transmitted vectors. Figure 3 shows that this theoretical formula is confirmed by simulation data.

### 3.3 Probability of error for a given minimum distance

In the sequel, $P_e(d_{\min})$ stands for the probability of error, given that the minimum distance is $d_{\min}$.(hence, it is a conditional probability). In [1] we have provided upper and lower bounds for this probability and we have shown that when signalling is BPSK, $P_e(d_{\min})$ is very close to the lower bound which is:

$$P_e(d_{\min}) = \frac{1}{2}\,\mathrm{erf}c\left(\frac{d_{\min}}{2\sigma}\right) \quad (21)$$

Figure 4 shows that this theoretical formula is in accordance with simulation data. Expressions for other modulations are given in [1].

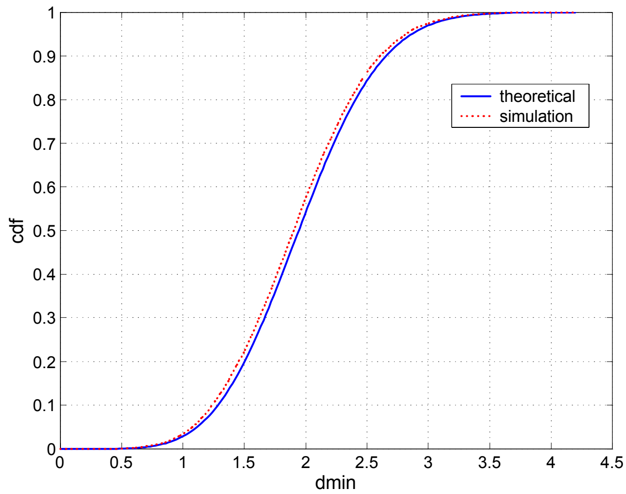


Figure 3: Verification of the theoretical formula for the cdf of $d_{\min}$. The simulation curve is obtained by averaging over 1000 random channel matrices.

### 3.4 Probability of Error

The probability of error can be written:

$$P_e = \int_0^\infty p(d_{\min}) P_e(d_{\min}) d d_{\min} \quad (22)$$

where $p(d_{\min})$ is the probability density function (pdf) of $d_{\min}$. Using integration by parts, we can write:

$$P_e = [F(d_{\min})P_e(d_{\min})]_0^\infty - \int_0^\infty F(d_{\min}) P_e^{'}(d_{\min}) d d_{\min} \quad (23)$$

where $P_e^{'}(d_{\min})$ is the derivative of $P_e(d_{\min})$ with respect to $d_{\min}$ and $F(d_{\min})$ the cdf of $d_{\min}$. Since $F(0) = 0$ and $P_e(\infty) = 0$, we have:

$$\boxed{P_e = -\int_0^\infty F(d_{\min}) P_e^{'}(d_{\min}) d d_{\min}} \quad (24)$$

This integral can be computed numerically. $F(d_{\min})$ is provided by equation 20, and:

$$P_e^{\prime}(d_{\min}) = -\frac{1}{2\sigma\sqrt{\pi}} \exp\left(-\left(\frac{d_{\min}}{2\sigma}\right)^2\right) \quad (25)$$

## 4 Reliability of simulation results

### 4.1 Relative precision

We recall that during simulations the probability of error is estimated by averaging over $N_H$ random channel

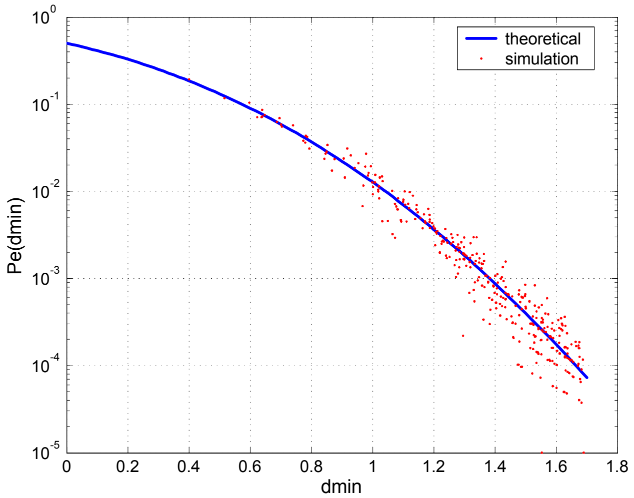


Figure 4: Verification of the theoretical formula for the conditional probability of error $P_e(d_{\min})$. The signal to noise ratio is $10dB$. Each simulation point correspond to a random channel matrix.

matrices $H$:

$$\widehat{P_e} = \frac{1}{N_H} \sum_H \widehat{P_e}(H) \tag{26}$$

Since the number $N_x$ of transmitted vectors for each channel matrix is usually large, we can consider that $\widehat{P_e}(H)$ is close to $P_e(d_{\min})$, where $d_{\min}$ is the minimum distance corresponding to matrix $H$. Hence:

$$var\left\{\widehat{P_e}\right\} = \frac{1}{N_H} var_{d_{\min}}\left\{\widehat{P_e}(d_{\min})\right\} \tag{27}$$

$$= \frac{\gamma}{N_H} \tag{28}$$

where:

$$\begin{aligned} \gamma &= var\left\{\widehat{P_e}(d_{\min})\right\} \\ &= \int_0^\infty p(d_{\min})\ (P_e(d_{\min}) - P_e)^2\ dd_{\min} \\ &= \left[F(d_{\min})(P_e(d_{\min}) - P_e)^2\right]_0^\infty \\ &- 2\int_0^\infty F(d_{\min})P_e^{'}(d_{\min})(P_e(d_{\min}) - P_e)^2\, dd_{\min} \\ &= (P_e)^2 \\ &- 2\int_0^\infty F(d_{\min})P_e^{'}(d_{\min})(P_e(d_{\min}) - P_e)^2\, dd_{\min} \end{aligned} \tag{29}$$

where $F(d_{\min})$, $P_e^{'}(d_{\min})$, $P_e(d_{\min})$ and $P_e$ are given by equations 20, 25, 21 and 24. Finally, the true relative precision, defined as the ratio between the standard deviation of the estimation and the true value is:

$$\rho = \frac{\sqrt{\gamma}}{P_e} \tag{30}$$

Figure 5 shows the relative precision with respect to the SNR. We can see that for an SNR larger than 12dB, the validity of simulation results is questionable.

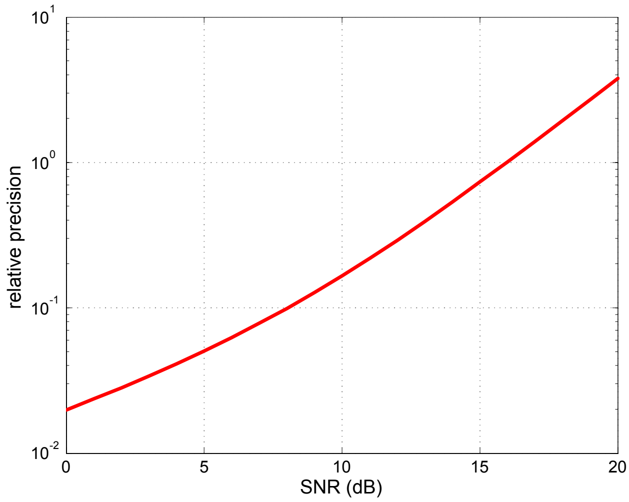


Figure 5: Relative precision with respect to SNR, for $N_H = 1000$.

## 4.2 Illustration

First of all, let us check our theoretical results. Figure 6 shows the experimental and theoretical estimations of the probability of error. The theoretical estimation is provided by equation 24 (with equations 20 and 25). The simulation results are obtained by averaging over 1000 random matrices $H$ and $2 \times 10^5$ vectors transmitted each $H$. The theoretical curve minus the standard deviation (square root of Eq. 29) is also shown.

In the previous subsection, we have seen that with $N_H = 1000$, the quality of the simulation results is questionable for an SNR larger than 12dB. This is confirmed by the fact that the simulation results clearly diverge from the theoretical curve above 12dB.

It is the theoretical results which are the most reliable. Indeed, if we increase the number of random channel matrices in the simulation process, the simulation curve converges to the theoretical curve. Furthermore, note that the theoretical curve is obtained in less than one second with a non optimized Matlab program, while days of computations are required to obtain the same results by simulation.

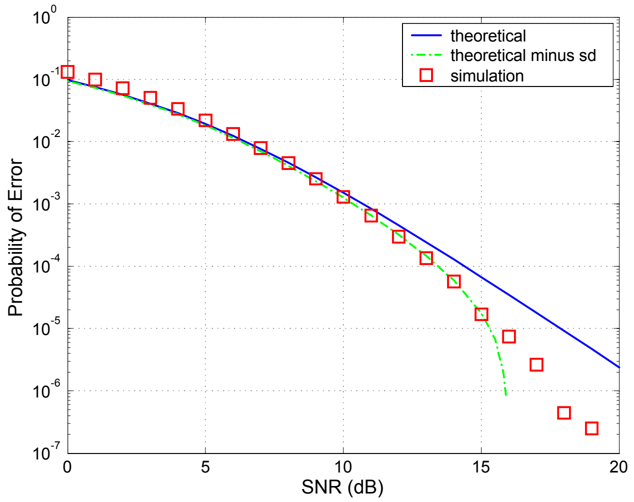


Figure 6: Probability of error with respect to SNR. Comparison of theoretical and simulation results.

Figure 7 illustrates the theoretical formula for the probability of error (Eq. 22). By analyzing the formula, it is easy to see that the probability of error is the surface under the curve labelled *product*, which represents $P_e(d_{\min})\, p\,(d_{\min})$ (this curve has been drawn up to a scale factor for easier visualization). We can note that the values of $d_{\min}$ which most contribute to the errors have a very low probability. Hence, if classical simulation is used, the probability of error will usually be **under-estimated**, unless a very large number of matrices $H$ is simulated (but this implies huge computation time).

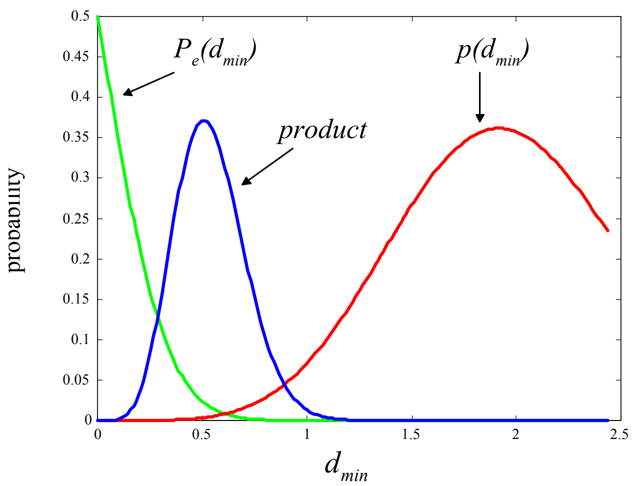


Figure 7: Illustration of the theoretical formula of the probability of error (SNR=15dB)

# 5 Conclusion

In this paper we have provided a fast and efficient method to predict the error rate in MIMO transmission systems and have used the obtained theoretical formula to explain why simulation results are not reliable at high SNR. Furthermore, we have provided a way to compute the relative precision of the simulation results. This formula is helpful to know which simulation results are reliable and which are not. With regards to these new results, it can be mentioned that the reliability of some simulation curves provided in the literature about MIMO transmissions is highly questionable.

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