Statistical comparison between $\max\text{-}d_{\min}$, max-SNR and MMSE precoders

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***Abstract*— In this study, we considered a MIMO system with available channel state information at the transmitter side and compared three known precoders based on different and pertinent criteria: the post processing signal-to-noise ratio (max-SNR), the mean square error between transmitted and received symbols (MMSE) and the minimum distance of the received constellation ($\max\text{-}d_{\min}$). We demonstrate that, alike the max-SNR and contrarily to the MMSE, the $\max\text{-}d_{\min}$ gives the maximum diversity order for a Rayleigh channel. An upper-bound of the MMSE diversity order is given. On the other hand, $d_{\min}$ of the three precoders will be statistically studied as a function of the number of antennas. In addition, BER simulations showed that the best precoder is the $\max\text{-}d_{\min}$.**

I. Introduction

Further to the constant evolution of wireless communications, the use of Multiple-Input Multiple-Output (MIMO) systems may enable one to improve a communication link within a scattering environment like an indoor WAN [2]. For example, spatial multiplexing (SM) is a simple solution to significantly improve the spectral efficiency by demultiplexing different symbols over the transmit antennas [2], [9]. However, other methods have been developed to enhance the transmission reliability evaluated through the bit error rate (BER). But, BER is affected by some criteria, *e.g.* the signal-to-noise ratio (SNR) or the distance between two received symbols.

In this study, on available channel state information (CSI) condition at both sides pertinent criteria are optimized by three linear couples of precoder and decoder. Among them, the first precoder maximizes the SNR (denoted max-SNR) upon reception by performing the MRT (Maximum- Ratio-Transmission) and MRC (Maximum-Ratio-Combining) in order to transmit a single symbol along the most favorable channel direction [8]. The second precoder minimizes the mean square error (MMSE) between the transmitted and the received symbols [6], [7]. These two schemes decouple the MIMO system matrix into independent subchannels. In this case, single symbol detection is performed on each substream. For the max-SNR scheme only one substream is used. The third precoder, which was recently described in [1], is based on the maximization of the minimum Euclidean distance between two received symbols denoted, here, $\max\text{-}d_{\min}$. Note that, unlike the max-SNR and MMSE cases, the received constellation is not the same as that of the transmit constellation. An ML symbol vector decision is then used for the $\max\text{-}d_{\min}$ precoder.

II. Channel model

At first, let us consider a MIMO system composed of n_R receive- and n_T transmit- antennas with two independent data streams. For a Rayleigh flat fading MIMO channel including a precoder matrix, $\mathbf{F}$, and a decoder matrix, $\mathbf{G}$, but no delay spread, the basic system model is expressed as:

$$\mathbf{y} = \mathbf{GHFs} + \mathbf{Gn} \tag{1}$$

where $\mathbf{H}$ is the $n_R \times n_T$ channel matrix, $\mathbf{F}$ is the $n_T \times 2$ precoder matrix, $\mathbf{G}$ is the $2 \times n_R$ decoder matrix, $\mathbf{s}$ is the 2×1 transmitted vector symbol, and $\mathbf{n}$ is the $n_R \times 1$ additive Gaussian white noise vector (AGWN). Let us assume $m = \text{rank}(\mathbf{H}) = \min(n_T, n_R) \geq 2$ and [1]

$$E[\mathbf{ss}^*] = \mathbf{I}_2, \;\; E[\mathbf{nn}^*] = \sigma_n^2 \mathbf{I}_{n_R}, \;\; \text{and} \;\; E[\mathbf{sn}^*] = \mathbf{0}. \tag{2}$$

Under the CSI condition at the transmitter side, a simplified representation of (1) can be performed by using the following decompositions $\mathbf{F} = \mathbf{F}_v\mathbf{F}_d$ and $\mathbf{G} = \mathbf{G}_d\mathbf{G}_v$. The unitary matrices, $\mathbf{G}_v$ and $\mathbf{F}_v$, based on the singular value decomposition (SVD) of $\mathbf{H}$ diagonalize the channel and reduce the dimensions to 2. Then, the received vector can be expressed as:

$$\mathbf{y} = \mathbf{G}_d\mathbf{H}_v\mathbf{F}_d\mathbf{s} + \mathbf{G}_d\mathbf{n}_v \tag{3}$$

where $\mathbf{H}_v = \mathbf{G}_v\mathbf{HF}_v$ is the virtual diagonal channel, and $\mathbf{n}_v = \mathbf{G}_v\mathbf{n}$ is the virtual AGWN with variance σ_n^2. Moreover, the symbols decision is a maximum likelihood rule and is unaffected by the linear decoder, $\mathbf{G}_d$, chosen equal to the identity matrix. The matrix, $\mathbf{H}_v = \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2})$, corresponds to two independent sub-channels where λ_1 and λ_2 are the two highest eigenvalues of $\mathbf{HH}^*$ ($\lambda_1 \geq \lambda_2$).

In order to simplify the statistical study, the two-dimension system can be rewritten without loss of generality by changing

[1] $\mathbf{I}_n$ is the identity matrix $n \times n$, $.^*$ is the transpose conjugate, $||.||_F$ is the Frobenius norm, $\mathcal{C}$ is the modulation alphabet, and $\mathcal{CN}(0,1)$ is the zero-mean and unit-variance complex normal distribution.

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from Cartesian to polar coordinates:

$$\left\{ \begin{array}{l} \lambda_1 = (\rho\cos\gamma)^2 \\ \lambda_2 = (\rho\sin\gamma)^2 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \gamma = \arctan\sqrt{\frac{\lambda_2}{\lambda_1}} \\ \rho = \sqrt{\lambda_1+\lambda_2} \end{array} \right. \quad (4)$$

So, the virtual channel can be written as :

$$\mathbf{H}_v = \rho \begin{pmatrix} \cos\gamma & 0 \\ 0 & \sin\gamma \end{pmatrix} \quad (5)$$

where ρ is a positive real parameter related to the channel gain, and γ is an angle linked to the eigenvalues ratio. It is worth noting that the virtual channel is totally defined by ρ and γ and that $\pi/4 \geq \gamma > 0$ ($\lambda_1 \geq \lambda_2 > 0$). Moreover, a small γ means that the first sub-channel is privileged ($\lambda_1 \gg \lambda_2$), whereas a value close to $\pi/4$ indicates two equivalent sub-channels ($\lambda_1 \simeq \lambda_2$). Now, all the precoders can be designed on using γ and ρ. The comparison of performances is simplified thanks to this change of variables: the study will investigate the precoders performances by using the parameter γ related to the dispersion of the singular values of the channel matrix.

III. Optimized criteria

We note the average total transmit power E_T and define the SNR as E_T/σ_n^2. Under the power constraint $||\mathbf{F}_d||_F^2 = E_T$, the matrix $\mathbf{F}_d$ is the result of the optimization of the three different criteria by:

1) either maximizing the post processing SNR. Only one symbol is transmitted along the singular vectors associated to the highest singular value $\sqrt{\lambda_1}$ [8]:

$$\mathbf{F}_d^{(\text{max-SNR})} = \sqrt{E_T}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (6)$$

As a result, the max-SNR uses only the first virtual sub-channel and always neglects the second eigenvalue.

2) or minimizing the MSE [6], [7] defined by:

$$\text{MSE}(\mathbf{F}_d, \mathbf{G}_d) = E[(\mathbf{y}-\mathbf{s})^*(\mathbf{y}-\mathbf{s})] \quad (7)$$

The solution is a diagonal matrix $\mathbf{F}_d^{(\text{MMSE})} = \text{diag}(f_1, f_2)$ with the following eigen-mode power allocation:

$$f_i^2 = \left\{ \begin{array}{ll} \frac{\sigma_n}{\sqrt{\lambda_i}}\left(\Psi - \frac{\sigma_n}{\sqrt{\lambda_i}}\right) & \text{if } \Psi > \frac{\sigma_n}{\sqrt{\lambda_i}} \\ 0 & \text{otherwise} \end{array} \right. , i = \{1,2\} \quad (8)$$

$$\text{with } \Psi = \frac{E_T + \sum_{i=1}^{b_\Psi} \frac{\sigma_n^2}{\lambda_i}}{\sum_{i=1}^{b_\Psi} \frac{\sigma_n}{\sqrt{\lambda_i}}} \quad (9)$$

where b_Ψ corresponds to the number of the sub-channels in use. Depending on the sub-channel SNR ($\frac{\lambda_i}{\sigma_n^2}$), the precoder can either transmit all the power to the first sub-channel and is, then, equivalent to the max-SNR or transmits one symbol to each of the two sub-channels.

By taking into account (4), this solution can also be written for $i = \{1,2\}$ as:

$$\text{if } \gamma \leq \gamma_{\text{MMSE}}(\Phi),\ f_1 = \sqrt{E_T} \text{ and } f_2 = 0 \quad (10a)$$

$$\text{else} \quad f_i^2 = \frac{E_T}{\Phi}\frac{\Phi\tan^{3-i}\gamma + (-1)^i(\tan\gamma - 1)(1+\tan^2\gamma)}{\tan\gamma(1+\tan\gamma)}$$
$$\text{for} \quad i = \{1,2\} \quad (10b)$$

where $\Phi = E_T\rho^2/\sigma_n^2$ is the received SNR and $\gamma_{\text{MMSE}}(\Phi) = \arctan(x_0)$ where x_0 is the unique real zero of $x^3 - x^2 + (1+\Phi)x - 1$. It is worth noting that MMSE$_{limit}$ of concern hereafter is the one defined as the precoder MMSE when the SNR is high: this precoder always uses the two subchannels and $\gamma_{\text{MMSE}}(\Phi) \to 0$. We obtain the following eigen-mode power allocation :

$$f_1^2 = E_T\frac{\tan\gamma}{1+\tan\gamma} \quad (11)$$

$$f_2^2 = E_T\frac{1}{1+\tan\gamma} \quad (12)$$

Note that, the MMSE solution at high SNR compensates the weakest mode ($f_2^2 > f_1^2$).

3) or maximizing the minimum Euclidian distance $d_{\min}$ defined by:

$$d_{\min}(\mathbf{F}_d) = \min_{\mathbf{s}_k,\mathbf{s}_l \in \mathcal{C}^2, \mathbf{s}_k \neq \mathbf{s}_l} \|\mathbf{H}_v\mathbf{F}_d(\mathbf{s}_k - \mathbf{s}_l)\|. \quad (13)$$

This precoder always uses two symbols and for a 4-QAM an analytical expression of the precoder can be found [1]:

$$\mathbf{F}_d^{(\text{max-}d_{\min})} = \left\{ \begin{array}{ll} \mathbf{F}_{r1} & \text{for } 0 < \gamma \leq \gamma_0 \\ \mathbf{F}_{octa} & \text{for } \pi/4 > \gamma \geq \gamma_0 \end{array} \right. \quad (14)$$

with

$$\mathbf{F}_{r1} = \sqrt{E_T}\begin{pmatrix} \sqrt{\frac{3+\sqrt{3}}{6}} & \sqrt{\frac{3-\sqrt{3}}{6}}e^{i\pi/12} \\ 0 & 0 \end{pmatrix} \quad (15a)$$

and

$$\mathbf{F}_{octa} = \frac{\sqrt{E_T}}{2}\begin{pmatrix} \cos\psi & 0 \\ 0 & \sin\psi \end{pmatrix}\begin{pmatrix} \sqrt{2} & 1+i \\ -\sqrt{2} & 1+i \end{pmatrix} \quad (15b)$$

where

$$\psi = \arctan\frac{\sqrt{2}-1}{\tan\gamma} \quad (16)$$

and $\gamma_0 = \arctan\sqrt{\frac{3\sqrt{3}-2\sqrt{6}+2\sqrt{2}-3}{3\sqrt{3}-2\sqrt{6}+1}} \simeq 17.28°$ is a SNR-independent threshold contrarily to the MMSE one. Note that, the precoder $\mathbf{F}_{r1}$ is similar to the max-SNR because the symbols s_1 and s_2 are mixed and are only transmitted on the highest SV $\sigma_1 = \sqrt{\lambda_1}$ of the channel.

To get a spectral efficiency equals to 4 bits/s/Hz we set the modulations of the three precoders at 4-QAM or 16-QAM for 2 or 1 used subchannel, respectively.

IV. Performance analysis

A. *full diversity of the* max-$d_{\min}$ *precoder*

The max-SNR is known to achieve the full diversity [4]. By using a similar development alike the one reported in [5, pp.99-100] for the max-SNR, the proof of the full diversity of the max-$d_{\min}$ is straightforward. The starting point is the use of the union bound of the vector-symbol error probability (SEP) which depends directly of the minimum distance when the ML detection rule is applied, and the following lower bound of the minimum distance of the max-$d_{\min}$ precoder:

$$\frac{E_T d_{\min_1}^2 \|\mathbf{H}\|^2}{m} \le E_T d_{\min_1}^2 \lambda_1 \le d_{\min}^2(\text{max-}d_{\min}) \quad (17)$$

where $E_T d_{\min_1}^2 \lambda_1$ is the square minimum distance of the max-SNR at the receiver side, with $d_{\min_1}$ the minimum distance of the max-SNR transmit constellation. Like in the max-SNR diversity order proof [4], inequality (17) permits to obtain the diversity order $n_T \times n_R$ for i.i.d. Rayleigh channel. Thus, the diversity order of the max-$d_{\min}$ is $n_T \times n_R$.

In an other way, $\mathbf{F}_{octa}$ in (14) is a non-diagonal matrix that brings transmit diversity to achieve full diversity when the precoder is associated with an ML detector, while with the MMSE solution the precoder matrix is diagonal and doesn't bring diversity in the simplified equivalent representation (3). The proof that the MMSE solution doesn't achieve the full diversity is presented in the next subsection.

B. MMSE diversity order

A high SNR means that the MMSE precoder always uses the two sub-channels ($\gamma_{\text{MMSE}}(\infty) \to 0$) and f_i are given by (11). The SEP can be considered as the mean of the two parallel sub-channel probabilities. Consequently, the worst sub-channel penalizes the global SEP and determinates the diversity order. Let us consider the second sub-channel received-SNR η_2 and its upper-bound ($\gamma < \pi/4$):

$$\eta_2 = \lambda_2 \frac{E_T}{\sigma_n^2} \frac{1}{1+\tan\gamma} \le \lambda_2 \text{SNR}. \quad (18)$$

Let us also consider the matrix $\widehat{\mathbf{H}}$ defined as $\mathbf{H}$ after a column- or row-reduction so that the minimum size is $m-1$. Thus the $m-1$ eigenvalues $\widehat{\lambda}_i$ of $\widehat{\mathbf{H}}$ are upper- and lower-bounded as follows ([3, p. 449]):

$$\lambda_1 \ge \widehat{\lambda}_1 \ge \lambda_2 \ge \widehat{\lambda}_2 \ge \cdots \ge \widehat{\lambda}_{m-1} \ge \lambda_m \quad (19)$$

and, consequently,

$$\eta_2 \le \widehat{\lambda}_1 \text{SNR}. \quad (20)$$

The MMSE second sub-channel SEP can be upper-bounded by the max-SNR performances in the case of channel $\widehat{\mathbf{H}}$ with $n_T \times n_R - \max(n_T, n_R)$ i.i.d. Gaussian elements. The MMSE diversity order can be then upper-bounded by $n_T \times n_R - \max(n_T, n_R)$.

C. Received $d_{\min}$ comparison

Figure 3 shows the BERs of the three precoders when $n_T = n_R = 3$ and evidences different diversity orders (asymptote slope at high SNR). The precoder max-$d_{\min}$ has a gain of 2 and 4 dB on max-SNR and MMSE, respectively. The reason why max-$d_{\min}$ performs better than max-SNR and MMSE can be evidenced by analyzing the influence of γ. Indeed, ρ corresponds to a scale factor which is the same for the three precoders, and thus γ rules the power allocation strategy. As a result, the received $d_{\min}$ of max-SNR and MMSE precoders with respect to the optimized $d_{\min}$ depends only on γ; the ratios are defined as:

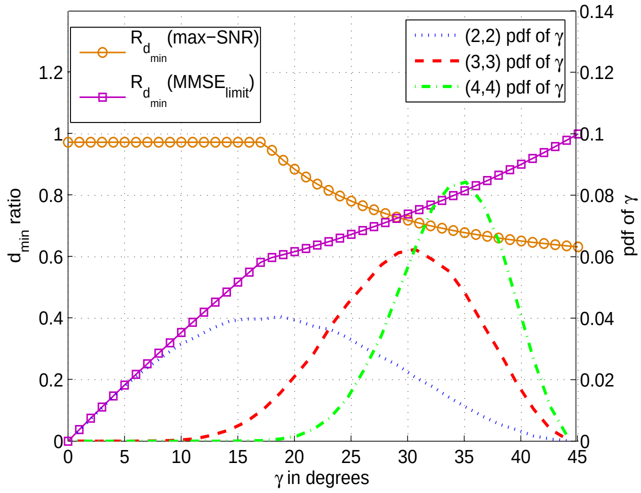

Fig. 1. Degradation of the $d_{\min}$ with respect to the optimized $d_{\min}$ as a function of γ for max-SNR and MMSE precoders together with pdfs of γ for $n_T = n_R = \{2, 3, 4\}$. Note *i)* the three different areas in ratio behaviors, *ii)* the shift to the right induced by increasing the number of antennas beneficial to the MMSE precoder.

$$\begin{aligned} \text{R}_{d_{\min}}(\text{max-SNR}) &= d_{\min}(\mathbf{F}_d^{(\text{max-SNR})})/d_{\min}(\mathbf{F}_d^{(\text{max-}d_{\min})}) \\ \text{R}_{d_{\min}}(\text{MMSE}_{\text{limit}}) &= d_{\min}(\mathbf{F}_d^{(\text{MMSE}_{\text{limit}})})/d_{\min}(\mathbf{F}_d^{(\text{max-}d_{\min})}) \end{aligned} \quad (21)$$

These two ratios of $d_{\min}$ are less than 1.

Figure 1 plots the $d_{\min}$ ratios

$\text{R}_{d_{\min}}(\text{max-SNR})$ and $\text{R}_{d_{\min}}(\text{MMSE}_{\text{limit}})$ together with the superimposition of the simulated pdfs of γ for $n_T = n_R = \{2, 3, 4\}$. It is worth noting that the pdfs are shifted towards the right-hand side with the number of antennas increasing. That means that, statistically, the channel's singular values are increasingly close ($\sigma_2 \to \sigma_1$ or $\gamma \to \pi/4$) which supports the precoder $\mathbf{F}_{octa}$ (case $\gamma > \gamma_0$). Note the discontinuity of the $d_{\min}$ ratios at $\gamma = \gamma_0 = 17.28°$ due to the change of precoding structure of the max-$d_{\min}$ solution.

We observe three different areas in the behaviors of $\text{R}_{d_{\min}}(\text{max-SNR})$ and $\text{R}_{d_{\min}}(\text{MMSE}_{\text{limit}})$: $0° \le \gamma \le 17°$ where MMSE is the worst precoder, $17° \le \gamma \le 29°$ characterized by a decrease of $\text{R}_{d_{\min}}(\text{max-SNR})$ concomitant with an increase of $\text{R}_{d_{\min}}(\text{MMSE}_{\text{limit}})$, $29° \le \gamma \le 45°$ where max-SNR becomes the worst precoder. Thus, the performances of precoders according to n_T and n_R can be qualitatively assessed from the pdf of γ; moreover, whenever the diversity order is also considered, it leads to the selection of the precoder the most suited to a given material configuration.

D. Performance comparisons in term of BER

We simulated the bit error rate (BER) of the three precoders for $n_T = n_R = \{2, 3, 4\}$ and 10^4 random matrices $\mathbf{H}$ with i.i.d. $\mathcal{CN}(0, 1)$ entries. One should note that, for a fixed number of transmit and receive antennas, the diversity order is

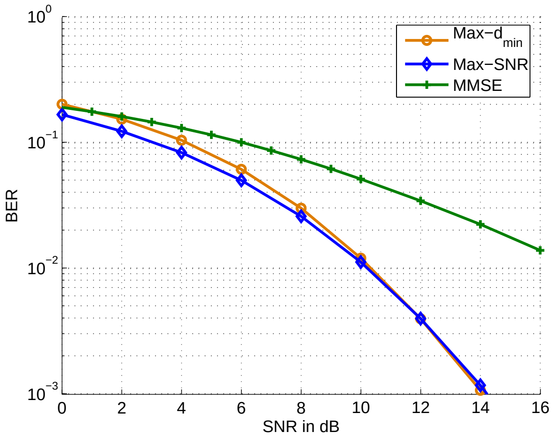

Fig. 2. BER simulations for $n_T = n_R = 2$ versus SNR

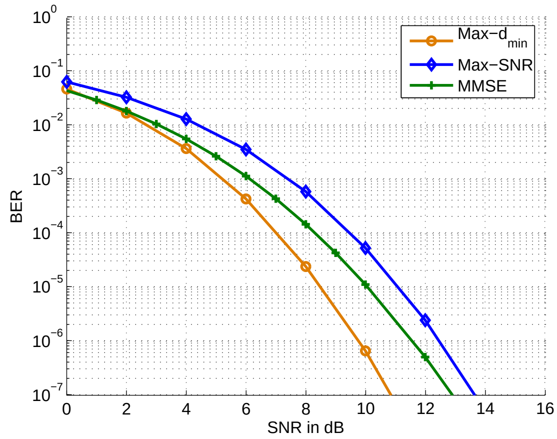

Fig. 4. BER simulations for $n_T = n_R = 4$ versus SNR

maximized when the antennas are symmetrically distributed on both sides. Thus, we will restrict BER simulations to symetric MIMO systems with $n_T = n_R$.

In Fig.2, BERs for $n_T = n_R = 2$ show that $\max$-SNR and $\max$-$d_{\min}$ precoders are equivalent at high SNR with a diversity order equal to 4. The diversity order from the MMSE precoders being about 1 cannot be compared to the others.

Fig.3 presents the simulations for $n_T = n_R = 3$; the gain provided by the $\max$-$d_{\min}$ is about 2 dB with respect to the $\max$-SNR. The MMSE is equivalent to $\max$-SNR when the SNR is less than 8 dB. Otherwise, the difference in diversity order, 9 against $9-3=6$ (or less), strongly increases the gap.

Fig.4 illustrates the simulation for $n_T = n_R = 4$ and shows that the $\max$-$d_{\min}$ remains the best precoder with gains of 2 dB and 3 dB on the MMSE and $\max$-SNR, respectively. The MMSE precoder is better than the $\max$-SNR for the investigated range of SNRs though the diversity order is smaller than the maximum diversity order: this drawback is compensed by the coding gain of the MMSE. Contrarily to the two others, the $\max$-SNR design does not use λ_2 and is penalized by an increasing number of antennas.

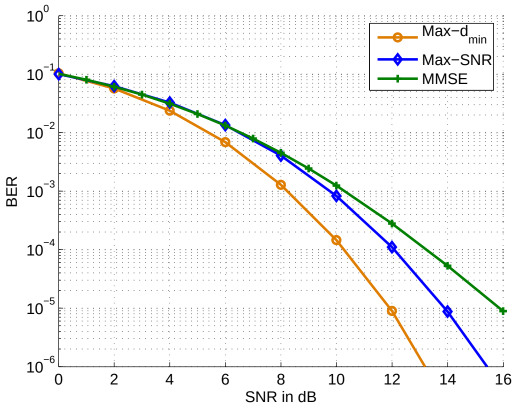

Fig. 3. BER simulations with $n_T = n_R = 3$ versus SNR for $\max$-SNR, MMSE and $\max$-$d_{\min}$ precoders for an uncorrelated Rayleigh channel

V. Conclusion

This study compared two well-known precoders, $\max$-SNR and the MMSE with a third and new one, $\max$-$d_{\min}$. Focus was placed on the diversity order. We showed that the maximum diversity order $n_T \times n_R$ was provided by the $\max$-$d_{\min}$ precoder, alike the $\max$-SNR, contrarily to the MMSE. Indeed, we demonstrated that the MMSE diversity order is smaller or equal to $n_T \times n_R - \max(n_T, n_R)$ for two independent data streams. However, at higher number of antennas, MMSE may enhance performances compared to those of the $\max$-SNR because the second weakest eigen sub-channel becomes increasingly significant with the number of antennas. The $\max$-SNR always discards this subchannel while MMSE take it into account. The $\max$-$d_{\min}$ precoder always produced the best performances in term of BER with the maximum diversity order.

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