Unbounded viscosity solutions of Hamilton-Jacobi equations with mechanical Hamiltonian in Wasserstein space

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In this talk, we discuss the value function arising in classical mechanics in the space of probability measures endowed with the 2−Wasserstein distance. The value function has the following form, analogous to the classical Euclidean case:

$$\forall (t,\mu)\in[0,T]\times\mathscr{P}_2(\mathbb{R}^N),\quad \vartheta(t,\mu)=\inf\Big\{\int_0^t \mathsf{L}(s,\sigma_s,\sigma_s')\mathrm{d}s+\ell(\sigma_0)\,:\,\sigma_t=\mu,\ \sigma\in\mathscr{AC}^2([0,t];\mathscr{P}_2(\mathbb{R}^N)\Big\},$$

where $\ell:\mathscr{P}_2(\mathbb{R}^N)\to\mathbb{R}$ is an initial cost and the Lagrangian L is defined as (with $\mathscr{W}$ a potential)

$$\forall\, t\in[0,T],\quad \forall(\mu,\eta)\in\mathfrak{T}\mathscr{P}_2(\mathbb{R}^N),\quad \mathsf{L}(t,\mu,\eta):=\frac{1}{2}|\eta|_\mu^2-\mathscr{W}(t,\mu).$$

As in the classical Euclidean case, we aim to show that the value function is the unique solution to a suitable Hamilton-Jacobi equation of the form

$$\begin{cases}\partial_t u(t,\mu)+\mathsf{H}(t,\mu,D_\mu u)=0, & \forall(t,\mu)\in(0,T)\times\mathscr{P}_2(\mathbb{R}^N),\\ u(0,\mu)=\ell(\mu),\end{cases}$$

where the Hamiltonian H is defined as

$$\mathsf{H}(t,\mu,p_\mu)=\sup_{\eta\in\mathfrak{T}_\mu\mathscr{P}_2(\mathbb{R}^N)}\Big\{p_\mu(\eta)-\mathsf{L}(t,\mu,\eta)\Big\}.$$

This problem can be seen as the limiting case of the "perturbed" problem with perturbed Lagrangian defined as:

$$\forall\, t\in[0,T],\quad \forall(\mu,\eta)\in\mathfrak{T}\mathscr{P}_2(\mathbb{R}^N),\quad \mathsf{L}(t,\mu,\eta):=\frac{1}{2}|\eta-\kappa\nabla_\mu\mathscr{E}(\mu)|_\mu^2-\mathscr{W}(t,\mu),$$

where $\mathscr{E}:\mathscr{P}_2(\mathbb{R}^N)\to\mathbb{R}\cup\{\infty\}$ the relative entropy function and $\kappa>0$.

Well-posedness of the Hamilton-Jacobi equation associated with the perturbed problem is well established in the literature under suitable assumptions. In contrast, proving well-posedness for the corresponding limiting problem ($\kappa=0$) using the same techniques turned out to be significantly more challenging and remains an open question. In this talk, we address the well-posedness of the limiting Hamilton-Jacobi equation by introducing a unified framework of viscosity solutions that allows to treat both the perturbed and limiting cases. More precisely, we

establish uniqueness via a comparison principle and show that the value function is indeed the unique viscosity solution to the limiting problem through dynamic programming arguments. The class of test functions we employ is closely related to that used in the perturbed setting. The main contribution of this work is to demonstrate that the limiting case can be treated within essentially the same viscosity-solution framework as the perturbed case.